THE COMBINATORIAL STRUCTURE OF (m, n)-CONVEX SETS

BY

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ABSTRACT

Let S be a closed subset of a Hausdorff linear topological space, S having no isolated points, and let $c_s(m)$ denote the largest integer n for which S is (m,n)-convex. If $c_s(k) = 0$ and $c_s(k+1) = 1$, then

$$c_s(m) = \sum_{i=1}^{k} \binom{\left[\frac{m+k-i}{k}\right]}{2}.$$

Moreover, if T is a minimal m subset of S, the combinatorial structure of T is revealed.

1. Introduction

Throughout, the set S will be a subset of a Hausdorff linear topological space. Employing the terminology used by Guay and Kay [2], for integers m, n, we say that S is *(m,n)-convex* iff for each m distinct points of S, at least n of the $\binom{m}{2}$ possible segments determined by these points are in S. For convenience, when $1 \geqq m \geqq 0$, we say S is $(m,0)$-convex. Thus the definition of (m,n)-convex is meaningful for any $m \geqq 0$ and for $\binom{m}{2} \geqq n \geqq 0$. A set S is *exactly (m,n)-convex* iff S is (m,n)-convex, and not $(m,n+1)$-convex, and $c_s(m)$ will denote the unique integer n for which S is exactly (m,n)-convex.

For notational purposes, $\sigma(k,m)$ will represent the following summation:

$$\sigma(k,m) \equiv \sum_{i=1}^{k} \binom{\left[\frac{m+k-i}{k}\right]}{2}$$

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Finally, we will make use of the following familiar definitions:
For x, y in S, we say x *sees* y *via* S iff the corresponding segment $[x, y]$ lies in S. A subset T of S is *visually independent via* S iff for every x, y in T, $x \neq y$, x does not see y via S.

2. A formula for $c_s(m)$

For S a closed (p, q)-convex set having no isolated points, $q \geqq 1$, we are interested in the possible values which may be assumed by the sequence $(c_s(m)\colon m \geqq 2)$. Letting k denote the largest integer for which $c_s(k) = 0$, the following theorems reveal that $c_s(m)$ is uniquely determined by k for every m, and in fact $c_s(m) = \sigma(k, m)$.

THEOREM 1. *If S is a closed (m,n)-convex set, $n \geqq 1$, then S is exactly $(m_0, 1)$-convex for some $m_0 \geqq 2$.*

PROOF. Clearly S has at most j isolated points $z_1, z_2, \cdots, z_j$ where $j < m$. Letting $T = S \sim \{z_1, \cdots, z_j\}$, T is $(m-j, n)$-convex. Let m_0 denote the smallest positive integer for which $c_T(m_0) > 0$. If T is convex, the result is trivial, so without loss of generality assume $m_0 \geqq 3$. We will show that $c_T(m_0) = 1$. Since $c_T(m_0 - 1) = 0$, there is a visually independent subset $\{x_1, \cdots, x_{m_0 - 1}\}$ of T having $m_0 - 1$ members. Since x_1 is not an isolated point, there is an infinite net in $T \sim \{x_1\}$ converging to x_1. For some y in this net, $[y, x_i] \nsubseteq T$ for every i, $1 < i \leqq m_0 - 1$. (Otherwise, there would be a subnet converging to x_1, each point of which sees via S a particular x_{i_0}, and since T is closed, $[x_1, x_{i_0}]$ would lie in T, a contradiction.)

Thus $\{x_1, \cdots, x_{m_0 - 1}, y\}$ is a set with m_0 members for which only one of the corresponding segments lies in T. We conclude that $c_T(m_0) = 1$ and $c_s(m_0 + j) = 1$.

REMARK. It is interesting to note that if S is not closed, the result fails. (See Example 1 of this paper.)

THEOREM 2. *Let S be a closed set having no isolated points and with $c_s(k) = 0$, $c_s(k+1) = 1$ for some integer k. Then*

$$c_s(m) \leqq \sum_{i=1}^{k} \binom{\left[\frac{m+k-i}{k}\right]}{2} \equiv \sigma(k, m)$$

for every integer $m \geqq 0$.

PROOF. We exhibit an m member subset of S having at most $\sigma(k, m)$ corresponding segments in S. Select a k member visually independent subset of S, say $\{x_1, \cdots, x_k\}$. For $1 \leqq i \leqq k$, let N_i denote an infinite net in $S \sim \{x_i\}$ converging to x_i. Clearly since S is closed, there exist disjoint subnets M_i, $1 \leqq i \leqq k$, such that for w_i in M_i, v_j in M_j, $i \neq j$, $[w_i, v_j] \nsubseteq S$.

For each i, $1 \leqq i \leqq k$, let S_i be a set consisting of exactly $\left[\dfrac{m+k-i}{k}\right]$ distinct points from M_i. Clearly to each S_i there correspond at most

$$\binom{\left[\frac{m+k-i}{k}\right]}{2}$$

segments lying in S. Furthermore, letting j denote the smallest positive integer for which k divides $m - j$, $1 \leqq j \leqq k$, and

$$\text{card}\left(\bigcup_{i=1}^{k} S_i\right) = \sum_{i=1}^{k} \left[\frac{m+k-i}{k}\right] = \frac{j(m+k-j)}{k} + \frac{(k-j)(m-j)}{k} = m.$$

Thus we have an m member subset of S for which there are at most $\sigma(k,m)$ corresponding segments in S, and $c_m(s) \leqq \sigma(k, m)$.

Theorem 2 may be restated to include the case for S having isolated points, as we see in the following corollary.

COROLLARY 1. *If S is a closed set having j isolated points, with $c_s(k+j) = 0$ and $c_s(k+j+1) = 1$, then $c_s(m+j) \leqq \sigma(k, m)$.*

To obtain a formula for $c_s(m)$, it remains to show that $c_s(m) \geqq \sigma(k, m)$. The following lemma will be useful.

LEMMA 1. *If S is any set for which $c_s(k) = 0$ and $c_s(k+1) = 1$, then given any m member subset T of S, there is some point of T which sees via S at least $\left[\dfrac{m-1}{k}\right]$ of the remaining points in T.*

PROOF. If $k = 1$, S is convex and the result is trivial. If $m < k + 1$, then there is some m member subset of S having no corresponding segment in S, and again the result is trivial.

Let $2 \leqq k + 1 \leqq m$. For the moment, assume that T contains a k member visually independent subset R. Then each of the remaining $m - k$ points in T must see via S at least one member of R, and by the pigeon-hole principle, some point of R sees at least $\dfrac{m-k}{k}$ members of $T \sim R$ if k divides m, and $\left[\dfrac{m-k}{k}\right] + 1$ members of $T \sim R$ otherwise.

Now if k divides m, then

$$\frac{m-k}{k} = \frac{m}{k} - 1 = \left[\frac{m-1}{k}\right],$$

the desired result. Otherwise,

$$\left[\frac{m-k}{k}\right] + 1 = \left[\frac{m}{k}\right] = \left[\frac{m-1}{k}\right]$$

and the result is true.

If T contains at most a $(k-j)$ member visually independent subset, $1 \leqq j < k$, then an identical argument shows that some point of T sees via S at least $\left[\frac{m-1}{k-j}\right]$ of the remaining points in T, completing the proof.

It is easy to see that the result in Lemma 1 is best possible. (For example, let S be a union of k disjoint convex sets, none a singleton point.)

Using Lemma 1, we obtain a formula for $c_s(m)$.

THEOREM 3. *If S is any set for which* $c_s(k) = 0$ and $c_s(k+1) = 1$, *then* $c_s(m) \geqq \sigma(k, m)$.

PROOF. We assume that $2 \leqq k + 1 \leqq m$, for otherwise the result is trivial. The proof is by induction. Since $c_s(k+1) = 1 = \sigma(k, k+1)$, the result is true for $m = k + 1$.

Assume the result true for $m < n$ to prove for $m = n$. Let T be an n member subset of S. By the lemma, there is some point x in T which sees via S at least $\left[\frac{n-1}{k}\right]$ of the remaining points in T. We examine $T \sim \{x\}$. Since $T \sim \{x\}$ consists of $(n-1)$ points in S, by our induction hypothesis, there are at least $\sigma(k, n-1)$ corresponding segments in S. Thus there are at least $\sigma(k, n-1)$ $+ \left[\frac{n-1}{k}\right]$ segments in S corresponding to T.

However,

$$\sum_{i=1}^{k} \binom{\left[\frac{n-1+k-i}{k}\right]}{2} + \left[\frac{n-1}{k}\right] = \sum_{i=2}^{k+1} \binom{\left[\frac{n+k-i}{k}\right]}{2} + \left[\frac{n-1}{k}\right]$$

$$= \sum_{i=2}^{k+1} \binom{\left[\frac{n+k-i}{k}\right]}{2} + \binom{\left[\frac{n-1}{k}\right]+1}{2} - \binom{\left[\frac{n-1}{k}\right]}{2}$$

$$= \sum_{i=2}^{k+1} \binom{\left[\frac{n+k-i}{k}\right]}{2} + \binom{\left[\frac{n+k-1}{k}\right]}{2} - \binom{\left[\frac{n-1}{k}\right]}{2} = \sigma(k, n).$$

Thus there are at least $\sigma(k,n)$ segments in S corresponding to T, and the induction is complete. We conclude that $c_s(m) \geqq \sigma(k,m)$.

COROLLARY 1. *Let S be a closed (p,q)-convex set, $q \geqq 1$, S having no isolated points, and let k be the unique integer for which $c_s(k)=0$ and $c_s(k+1)=1$. Then $c_s(m)=\sigma(k,m)$ for every m.*

COROLLARY 2. *Let S be a closed (p,q) convex set, $q \geqq 1$, S having exactly j isolated points, and let k be the unique integer for which $c_s(k+j)=0$ and $c_s(k+j+1)=1$. Then $c_s(m+j)=\sigma(k,m)$.*

3. Minimal m subsets of S

Let S be an (m,n)-convex set. We say an m member subset T of S is a *minimal m subset of S* iff exactly $c_s(m)$ of the segments determined by T lie in S. The following lemma and theorem reveal the combinatorial structure of such a T.

LEMMA 2. *Let S be a closed set having no isolated points, with $c_s(k)=0$ and $c_s(k+1)=1$. If T is a minimal m subset of S, then no point of T sees more than $\left[\frac{m-1}{k}\right]$ of the remaining points of T. Moreover, T contains a descending chain of sets $\{T_j\}$ where each T_j is a minimal j subset of S, $1 \leqq j \leqq m$.*

PROOF. If some x in T sees via S more than $\left[\frac{m-1}{k}\right]$ of the points in $T \sim \{x\}$, then since $\sigma(k,m-1)+\left[\frac{m-1}{k}\right]=\sigma(k,m)=c_s(m)$, $T \sim \{x\}$ necessarily has fewer than $\sigma(k,m-1)$ corresponding segments in S. However $\sigma(k,m-1) = c_s(m-1)$, so this is impossible. We have a contradiction, and the first statement is proved.

By Lemma 1, some point x_1 in T sees via S at least $\left[\frac{m-1}{k}\right]$ points of $T \sim \{x_1\}$, so such an x_1 must see via S exactly $\left[\frac{m-1}{k}\right]$ points of $T \sim \{x_1\}$, and $T \sim \{x_1\}$ has exactly $c_s(m-1)$ corresponding segments in S. Thus $T \sim \{x_1\} \equiv T_{m-1}$ is a minimal $(m-1)$ subset of S. By induction it is easy to define a descending chain $\{T_j\}$ of subsets of T, $1 \leqq j \leqq m$, where each T_j is a minimal j subset of S.

REMARK. Lemma 2 may be suitably adapted in case S has isolated points.

THEOREM 4. *Let S be a closed set having no isolated points, with $c_s(k)=0$ and $c_s(k+1)=1$. If T is a minimal m subset of S, then the members of T may* *be partitioned into disjoint subsets* C_i, $1 \leqq i \leqq \left[\frac{m+k-1}{k}\right]$, *such that for* $1 \leqq i \leqq \left[\frac{m}{k}\right]$, *each* C_i *consists of exactly* k *visually independent points, and* $C_{[m/k]+1}$ *consists of* $m - k\left[\frac{m}{k}\right]$ *visually independent points.*

PROOF. The proof is by induction. If $1 \leqq m \leqq k$, then $T = C_1$, and the theorem is trivially satisfied. If $m = k + 1$, T necessarily contains a k member visually independent subset C_1. The remaining point in T yields C_2.

Assume the result true for m, $k + 1 < m < n$, to prove for $m = n$. Let j denote the smallest positive integer for which k divides $n - j$. Clearly $1 \leqq j \leqq k$. Using the procedure employed in the proof of Lemma 2, we may select points $x_1, x_2, \cdots, x_j$ in T such that for each i, x_i sees via S exactly $\left[\frac{n-i}{k}\right]$ points of $T \sim \{x_1, \cdots, x_{i-1}\}$, and $T \sim \{x_1, \cdots, x_i\}$ is a minimal $(n - i)$ subset of S, $1 \leqq i \leqq j$.

By Lemma 2, no point of T sees more than $\left[\frac{n-1}{k}\right]$ of the remaining points of T. Thus since $\left[\frac{n-1}{k}\right] = \left[\frac{n-i}{k}\right]$ for $1 \leqq i \leqq j$, every x_i sees no point of $\{x_1, \cdots, x_{i-1}\}$, and the set $\{x_1, \cdots, x_j\}$ is visually independent.

Since $T \sim \{x_1, \cdots, x_j\}$ is a minimal $(n - j)$ subset of S, by our induction hypothesis, this set has $\left[\frac{n-j+k-1}{k}\right] = \frac{n-j}{k} = \left[\frac{n-1}{k}\right]$ disjoint k member subsets, each one visually independent. These sets, together with $\{x_1, \cdots, x_j\}$ are seen to be the required sets.

If k divides n, then $j = k$ and we have $\left[\frac{n-1}{k}\right] + 1 = \frac{n}{k}$ k member sets. Otherwise we have $\left[\frac{n-1}{k}\right] = \left[\frac{n}{k}\right]$ disjoint k member sets and one $j = n - k\left[\frac{n}{k}\right]$ member set. This completes the induction and finishes the proof.

It is interesting to notice that if we do not require the set S to be closed, then for any $m > n \geqq 1$, there is a set which is (m, n)-convex and connected. Also, S may be constructed so that it is $(m_0, 0)$-convex for every $m_0 < m$, as Example 1 reveals.

EXAMPLE 1. Let T be a triangle in R^2, L one of its sides. For $m = n + 1$, choose points $x_1, \cdots, x_n$ on L with x_1 and x_n vertices of T. Let $S \equiv (T \sim L) \cup \{x_1, \cdots, x_n\}$. Then S is $(n + 1, n)$-convex. If $n = 1$, S is convex. (See Fig. 1.)

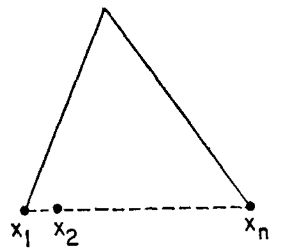

Fig. 1

Inductively, for $m = n + j$, $j \geqq 2$, join $j - 1$ copies of S as in Fig 2. (Note that two successive copies share all but two of their special x_i points.)

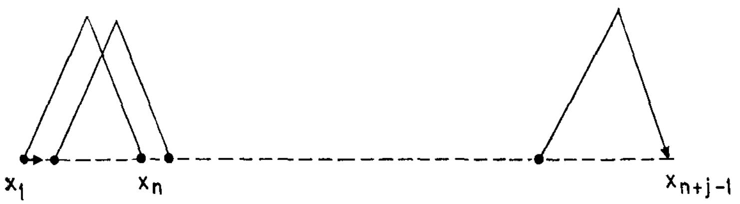

Fig. 2

Figure 2 may be altered by slicing off i peaks, $0 \leqq i \leqq j - 2$. The resulting figure S_i is exactly $(n + j, n)$-convex and $(m_0, 0)$-convex for every $m_0 < n + j$.

Moreover, n and j do not determine $c_s(m)$ for $m > n + j$. For $0 \leqq i \leqq j - 2$ and $m - (n + j) \geqq j - 2 - i$, each S_i set has a different value for $c_{s_i}(m)$, $m > n + j$ Of course $\sigma(n + j - 1, m)$ is still a lower bound for $c_s(m)$ by the proof of Theorem 3.

Although the question concerning the existence of an exactly (m, n)-convex set, $n > m \geqq 4$, is not completely solved, the existence of various (m, n)-convex sets can be verified by appropriately adapting the sets in Example 1. For instance, in Fig. 1, inserting j segments $[x_{i-1}, x_i]$ which alternate along L, we obtain an exactly $(n + 1, n + j)$-convex set for $2 \leqq n$, $1 \leqq j \leqq [n/2]$. Inserting j consecutive segments along L, we obtain an exactly $\left(n + 1, n + \binom{j+1}{2}\right)$-convex set for $1 \leqq j \leqq n - 1$.

Of course, if we require our set to be closed, the question is settled by the corollaries to Theorem 3. The construction of a closed, connected, exactly $(k + 1, 1)$-convex set is easy, $k \geqq 1$. Hence for $m \geqq 1$, $n \geqq 0$, there is a closed connected set which is exactly (m, n)-convex if and only if for some $1 \leqq k \leqq m$, $n = \sigma(k, m)$.

Similarly, if we allow isolated points, for $m_0 \geqq 1$, $n \geqq 0$, there is a closed, exactly (m_0, n)-convex set if and only if there exist some $j \geqq 0$ and some $1 \leqq k \leqq m$ such that $m_0 = m + j$ and $n = \sigma(k, m)$.

NOTE. The referee has pointed out that a paper soon to be published by J. Kaapke [1] offers an alternate approach to the proof of Theorem 1. Also, alternate proofs of Theorems 2 and 3 may be obtained from a theorem of Turán [3] and a formula in remark 2 of [2].

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